5. MATRICES

MATRIX *A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns) is called a matrix of order m by n, written as an $m \times n$ matrix.*

Such an array is enclosed by [ ] or ( ).

Each of the *mn* numbers constituting the matrix is called an *element* or an *entry* of the matrix.

Usually, we denote a matrix by a capital letter.

The plural of matrix is matrices.

Example (i) $A = \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 9 \end{bmatrix}$ is a matrix, having 2 rows and 3 columns.

Its order is 2×3 and it has 6 elements.

(ii) $B = \begin{bmatrix} 9 & 4 & \sqrt{2} & -1 \\ 1 & 8 & -3 & 2 \\ 6 & 0 & 5 & 7 \end{bmatrix}$ is a matrix, having 3 rows and 4 columns. Its order is 3×4 and it has 12 elements.

How to Describe a Matrix

In order to locate the position of a particular element of a matrix, we have to specify the number of the row and that of the column in which the element occurs.

An element occurring in the ith row and jth column of a matrix A will be called the (i, j)th element of A, to be denoted by a_{ij}.

In general, an $m \times n$ matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

EXAMPLE 1 *Consider the matrix* $A = \begin{bmatrix} 3 & -2 & 5 \\ 6 & 9 & 1 \end{bmatrix}$.

Clearly, the element in the 1st row and 2nd column is –2.

So, we write $a_{12} = -2$.

Similarly, $a_{11} = 3$; $a_{12} = -2$; $a_{13} = 5$; $a_{21} = 6$; $a_{22} = 9$ and $a_{23} = 1$.

EXAMPLE 2 *Construct a* 3×2 *matrix whose elements are given by* $a_{ij} = (i + 2j)$.

SOLUTION A 3×2 matrix has 3 rows and 2 columns.

In general, a 3×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2.}$$

Thus $a_{ij} = (i + 2j)$ for $i = 1, 2, 3$ and $j = 1, 2$.

$\therefore \quad a_{11} = (1 + 2 \times 1) = 3; \; a_{12} = (1 + 2 \times 2) = 5;$

$a_{21} = (2 + 2 \times 1) = 4; \; a_{22} = (2 + 2 \times 2) = 6;$

$a_{31} = (3 + 2 \times 1) = 5; \; a_{32} = (3 + 2 \times 2) = 7.$

Hence, $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}_{3 \times 2.}$

EXAMPLE 3 *Construct a* 2×3 *matrix whose elements are given by* $a_{ij} = \frac{1}{2}|5i - 3j|$.

SOLUTION A 2×3 matrix has 2 rows and 3 columns.

In general, a 2×3 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{3 \times 2.}$$

Thus, $a_{ij} = \frac{1}{2}|5i - 3j|$ where $i = 1, 2$ and $j = 1, 2, 3$.

$\therefore \quad a_{11} = \frac{1}{2}|5 \times 1 - 3 \times 1| = \frac{1}{2} \cdot |2| = \frac{1}{2} \times 2 = 1;$

$a_{12} = \frac{1}{2}|5 \times 1 - 3 \times 2| = \frac{1}{2} \cdot |5 - 6| = \frac{1}{2} \cdot |-1| = \frac{1}{2} \times 1 = \frac{1}{2};$

$a_{13} = \frac{1}{2}|5 \times 1 - 3 \times 3| = \frac{1}{2} \cdot |5 - 9| = \frac{1}{2} \cdot |-4| = \frac{1}{2} \times 4 = 2;$

$a_{21} = \frac{1}{2}|5 \times 2 - 3 \times 1| = \frac{1}{2} \cdot |10 - 3| = \frac{1}{2} \cdot |7| = \frac{1}{2} \times 7 = \frac{7}{2};$

$a_{22} = \frac{1}{2}|5 \times 2 - 3 \times 2| = \frac{1}{2} \cdot |10 - 6| = \frac{1}{2} \cdot |4| = \frac{1}{2} \times 4 = 2;$

$a_{23} = \frac{1}{2}|5 \times 2 - 3 \times 3| = \frac{1}{2} \cdot |10 - 9| = \frac{1}{2} \cdot |1| = \frac{1}{2} \times 1 = \frac{1}{2} \cdot$

Hence, $A = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ \frac{7}{2} & 2 & \frac{1}{2} \end{bmatrix}$.

EXAMPLE 4 *If a matrix has* 12 *elements, what are the possible orders it can have?*

SOLUTION We know that a matrix of order $m \times n$ has mn elements.

Hence, all possible orders of a matrix having 12 elements are (12×1), (1×12), (6×2), (2×6), (4×3) and (3×4).

EXERCISE 5A

1. If $A = \begin{bmatrix} 5 & -2 & 6 & 1 \\ 7 & 0 & 8 & -3 \\ \sqrt{2} & \frac{3}{5} & 4 & 3 \end{bmatrix}$ then write

   (i) the number of rows in A, (ii) the number of columns in A,

   (iii) the order of the matrix A, (iv) the number of all entries in A,

   (v) the elements a_{23}, a_{31}, a_{14}, a_{33}, a_{22} of A.

2. Write the order of each of the following matrices:

   (i) $A = \begin{bmatrix} 3 & 5 & 4 & -2 \\ 0 & \sqrt{3} & -1 & \frac{4}{9} \end{bmatrix}$ (ii) $B = \begin{bmatrix} 6 & -5 \\ \frac{1}{2} & \frac{3}{4} \\ -2 & -1 \end{bmatrix}$

   (iii) $C = [7 \quad -\sqrt{2} \quad 5 \quad 0]$ (iv) $D = [8 \quad -3]$

   (v) $E = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ (vi) $F = [6]$

3. If a matrix has 18 elements, what are the possible orders it can have?
4. Find all possible orders of matrices having 7 elements.
5. Construct a 3×2 matrix whose elements are given by $a_{ij} = (2i - j)$.
6. Construct a 4×3 matrix whose elements are given by $a_{ij} = \frac{i}{j}$.
7. Construct a 2×2 matrix whose elements are $a_{ij} = \frac{(i + 2j)^2}{2}$. **[CBSE 2002, '07]**
8. Construct a 2×3 matrix whose elements are $a_{ij} = \frac{(i - 2j)^2}{2}$. **[CBSE 2002]**
9. Construct a 3×4 matrix whose elements are given by $a_{ij} = \frac{1}{2}|-3i + j|$.

ANSWERS (EXERCISE 5A)

1. (i) 3 (ii) 4 (iii) 3×4 (iv) 12 (v) $a_{23} = 8,\ a_{31} = \sqrt{2},\ a_{14} = 1,\ a_{33} = 4,\ a_{22} = 0$

2. (i) (2×4) (ii) (3×2) (iii) (1×4) (iv) (1×2) (v) (3×1) (vi) (1×1)

3. $(18 \times 1), (1 \times 18), (9 \times 2), (2 \times 9), (6 \times 3), (3 \times 6)$ **4.** $(7 \times 1), (1 \times 7)$

5. $A = \begin{bmatrix} 1 & 9 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \\ 4 & 2 & \frac{4}{3} \end{bmatrix}$ **7.** $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

8. $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

Various Types of Matrices

ROW MATRIX *A matrix having only one row is known as a row matrix or a row vector.*

Examples (i) $A = [5 \ \ 18]$ is a row matrix of order 1×2.

(ii) $B = [2 \ \ \sqrt{5} \ \ -9 \ \ 0]$ is a row matrix of order 1×4.

COLUMN MATRIX *A matrix having only one column is known as a column matrix or a column vector.*

Examples (i) $A = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column matrix of order 3×1.

(ii) $B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is a column matrix of order 2×1.

ZERO OR NULL MATRIX *A matrix each of whose elements is zero is called a zero matrix or a null matrix.*

Examples The matrices $[0]$, $[0 \ \ 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices of order $(1 \times 1), (1 \times 2), (2 \times 2)$ and (2×3) respectively.

SQUARE MATRIX *A matrix having the same number of rows and columns is called a square matrix.*

A matrix of order $(n \times n)$ is called a *square matrix of order n* or an *n-rowed square matrix.*

A matrix of order $m \times n$, where $m \neq n$, is called a *rectangular matrix.*

Examples (i) The matrix $\begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ is a 2-rowed square matrix.

(ii) The matrix $\begin{bmatrix} 5 & 3 & 6 \\ 7 & \sqrt{2} & -4 \\ -9 & \frac{1}{3} & 0 \end{bmatrix}$ is a 3-rowed square matrix.

DIAGONAL ELEMENTS OF A MATRIX *Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix. Then, the elements a_{ij} for which $i = j$, are called the diagonal elements of A.*

Thus, the diagonal elements of $A = [a_{ij}]_{m \times n}$ are a_{11}, a_{22}, a_{33}, a_{44}, etc.

The line along which the diagonal elements lie is called the diagonal of the matrix.

Example Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{5} & \frac{5}{8} & 7 \\ 6 & -4 & \sqrt{2} \end{bmatrix}$

Then, the diagonal elements of A are:

$$a_{11} = 3, \ a_{22} = \frac{5}{8}, \ a_{33} = \sqrt{2}.$$

DIAGONAL MATRIX *A square matrix in which every nondiagonal element is zero is called a diagonal matrix.*

If $A = [a_{ij}]_{n \times n}$ be a diagonal matrix then $a_{ij} = 0$ when $i \neq j$ and we write it as

$A = \text{diag } [a_{11}, \ a_{22}, \ a_{33}, \ \ldots, \ a_{nn}]$.

Example Let $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then, A is a diagonal matrix.

We may write it as, $A = \text{diag } [6, \ 4, \ -2]$.

SCALAR MATRIX *A square matrix in which every nondiagonal element is zero and all diagonal elements are equal is known as a scalar matrix.*

Examples (i) $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ is a scalar matrix of order 2.

(ii) $B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix of order 3.

UNIT MATRIX *A square matrix in which every nondiagonal element is* 0 *and every diagonal element is* 1 *is called a unit matrix or an identity matrix.*

Thus, a square matrix $[a_{ij}]_{n\times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 & \text{when } i \neq j, \\ 1 & \text{when } i = j. \end{cases}$$

A unit matrix of order n will be denoted by I_n or simply by I.

Examples (i) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

(ii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

COMPARABLE MATRICES *Two matrices A and B are said to be comparable if they are of the same order, i.e., they have the same number of rows and the same number of columns.*

Example $A = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 & 0 \\ 1 & 4 & -9 \end{bmatrix}$ are comparable matrices, each being of order (2×3).

EQUAL MATRICES *Two matrices A and B are said to be equal, written as* $A = B$, *if they are of the same order and their corresponding elements are equal.*

EXAMPLE 1 *Find* $x,\ y,\ z$ *when* $\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix}$.

SOLUTION Since the corresponding elements of equal matrices are equal, we have

$$\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix} \Leftrightarrow x = 1,\ y = 5 \text{ and } z = 3.$$

EXAMPLE 2 *Find x, y, z, w when* $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$. **[CBSE 2013]**

SOLUTION We know that in equal matrices, the corresponding elements are equal.

$$\therefore \quad \begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$$\Leftrightarrow \quad x - y = -1,\ 2x - y = 0,\ 2x + z = 5 \text{ and } 3z + w = 13.$$

Solving the first two equations, we get $x = 1$ and $y = 2$.
Putting $x = 1$ in $2x + z = 5$, we get $z = 3$.
Putting $z = 3$ in $3z + w = 13$, we get $w = 4$.
$\therefore \quad x = 1, \quad y = 2, \quad z = 3$ and $w = 4$.

EXAMPLE 3 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. *Why?*

SOLUTION Since the given null matrices are not comparable, they are not equal.

Operations on Matrices

ADDITION OF MATRICES *Let A and B be two comparable matrices, each of order $(m \times n)$. Then, their sum $(A + B)$ is a matrix of order $(m \times n)$, obtained by adding the corresponding elements of A and B.*

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then

$$A + B = [a_{ij} + b_{ij}]_{m \times n}.$$

REMARK For two matrices A and B, the sum $(A + B)$ exists only when A and B are comparable.

Example 1 If $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ then A and B are matrices of order 2×2 and 2×3 respectively.
So, A and B are not comparable.
Hence, $A + B$ is not defined.

Example 2 Let $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 2 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$.
Clearly, each one of A and B is a 2×3 matrix.
So, A and B are comparable matrices.
$\therefore \quad A + B$ is defined.

$$\text{Now, } A + B = \begin{bmatrix} 5+4 & 0+(-3) & -2+(-6) \\ 3+(-1) & 2+0 & -7+4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & -8 \\ 2 & 2 & -3 \end{bmatrix}.$$

Some Results on Addition of Matrices

THEOREM 1 *Matrix addition is commutative,*
i.e., $A + B = B + A$ for all comparable matrices A and B.

PROOF Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then,

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$$
$$= [a_{ij} + b_{ij}]_{m\times n} \quad \text{[by the definition of addition of matrices]}$$
$$= [b_{ij} + a_{ij}]_{m\times n} \quad [\because \text{ addition of numbers is commutative}]$$
$$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n} = B + A.$$

Hence, $A + B = B + A$.

THEOREM 2 *Matrix addition is commutative,*
i.e, $(A + B) + C = A + (B + C)$ *for all comparable matrices A, B, C.*

PROOF Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$. Then,

$$(A + B) + C = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$$
$$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$$
$$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$$
$$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n}$$
$$[\because \text{ addition of numbers is associative}]$$
$$= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n}$$
$$= [a_{ij}]_{m\times n} + ([b_{ij}] + [c_{ij}]_{m\times n}) = A + (B + C).$$

Hence, $(A + B) + C = A + (B + C)$.

THEOREM 3 *If A is an $m \times n$ matrix and O is an $m \times n$ null matrix then*

$$A + O = O + A = A.$$

PROOF Let $A = [a_{ij}]_{m\times n}$ and $O = [b_{ij}]_{m\times n}$,
where $b_{ij} = 0$ for all suffixes i and j.

Then, $A + O = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$
$= [a_{ij} + 0]_{m\times n} \quad [\because \ b_{ij} = 0]$
$= [a_{ij}]_{m\times n} = A.$

$\therefore \quad A + O = A.$

Similarly, $O + A = A$.

Hence, $A + O = O + A = A$.

REMARK The null matrix O of order $m \times n$ is the additive identity in the set of all $m \times n$ matrices.

EXAMPLE 3 *Let* $A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix}$ *and* $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then verify that* $A + O = O + A = A$.

SOLUTION Clearly, each one of A and O is a matrix of order (2×3).

So, $(A + O)$ and $(O + A)$ are both defined.

$$\text{Now, } A + O = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3+0 & 5+0 & 4+0 \\ 1+0 & 2+0 & -3+0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A.$$

$$\text{And, } O + A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+3 & 0+5 & 0+4 \\ 0+1 & 0+2 & 0+(-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A.$$

Hence, $A + O = O + A = A$.

NEGATIVE OF A MATRIX *Let $A = [a_{ij}]_{m \times n}$. Then, the negative of A is the matrix $(-A) = [-a_{ij}]_{m \times n}$, obtained by replacing each element of A with its corresponding additive inverse. $(-A)$ is called the additive inverse of A.*

EXAMPLE 4 *If* $A = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$, *find* $(-A)$ *and verify that*

$$A + (-A) = (-A) + A = 0.$$

SOLUTION Clearly, we have:

$$(-A) = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix}$$

$$A + (-A) = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix} + \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3+(-3) & -2+2 & 0+0 \\ -5+5 & 7+(-7) & \sqrt{2}+(-\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

$$\text{And, } (-A) + A = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} + \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -3+3 & 2+(-2) & 0+0 \\ 5+(-5) & -7+7 & -\sqrt{2}+\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $A + (-A) = (-A) + A = O$.

SUMMARY

Laws of Addition on Matrices

(i) $A + B = B + A$ [Commutative law]
(ii) $(A + B) + C = A + (B + C)$ [Associative law]
(iii) $A + O = O + A = A$
(iv) $A + (-A) = (-A) + A = O$

SOLVED EXAMPLES

EXAMPLE 1 *Let* $A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix}$.

Verify that $A + B = B + A$.

SOLUTION Here, A is a 2×3 matrix and B is a 2×3 matrix. So, A and B are comparable.

Therefore, $(A + B)$ and $(B + A)$ both exist and each is a 2×3 matrix.

$$\text{Now, } A + B = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 3+(-2) & 5+3 \\ -1+2 & 0+6 & 4+(-1) \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}.$$

$$\text{And, } B + A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & -2+3 & 3+5 \\ 2+(-1) & 6+0 & (-1)+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}.$$

Hence, $A + B = B + A$.

EXAMPLE 2 *Let* $A = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$ *and* $C = \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$.

Verify that $(A + B) + C = A + (B + C)$.

SOLUTION Clearly, each one of the matrices A, B, C is a (3×2) matrix. So, $(A + B) + C$ and $A + (B + C)$ are both defined and each one is a 3×2 matrix.

$$\text{Now, } (A + B) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & -2+1 \\ 5+0 & 4+2 \\ 3+(-3) & 0+5 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix}.$$

$$\therefore \quad (A + B) + C = \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & -1+3 \\ 5+(-2) & 6+2 \\ 0+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}.$$

$$\text{Also, } (B + C) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 1+3 \\ 0+(-2) & 2+2 \\ -3+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix}$$

$$\therefore \quad A + (B + C) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1+7 & -2+4 \\ 5+(-2) & 4+4 \\ 3+(-2) & 0+11 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}.$$

Hence, $(A + B) + C = A + (B + C)$.

EXAMPLE 3 *Find the additive inverse of the matrix* $A = \begin{bmatrix} 2 & -5 & 0 \\ 4 & 3 & -1 \end{bmatrix}$.

SOLUTION Clearly, the additive inverse of the given matrix A is the matrix $-A$, given by

$$-A = \begin{bmatrix} -2 & -(-5) & 0 \\ -4 & -3 & -(-1) \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ -4 & -3 & 1 \end{bmatrix}.$$

SUBTRACTION OF MATRICES *If A and B are two comparable matrices then we define*

$$(A - B) = A + (-B).$$

EXAMPLE 4 *If* $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 8 & -4 \end{bmatrix}$, *find* $(A - B)$.

SOLUTION We have, $(-B) = \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix}$.

$$\therefore \quad (A - B) = A + (-B)$$

$$= \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-1) & -3+(-2) & 1+3 \\ 0+(-4) & 7+(-8) & -9+4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ -4 & -1 & -5 \end{bmatrix}.$$

Hence, $(A - B) = \begin{bmatrix} 1 & -5 & 4 \\ -4 & -1 & -5 \end{bmatrix}$.

EXAMPLE 5 *If* $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$, *then find the value of* $(a - 2b)$. **[CBSE 2014]**

SOLUTION Comparing the corresponding elements of given equal matrices, we have $a + 4 = 2a + 2$, $3b = b + 2$ and $a - 8b = -6$.

From these equations, we get $a = 2$ and $b = 1$.

$\therefore \quad (a - 2b) = (2 - 2 \times 1) = (2 - 2) = 0.$

EXAMPLE 6 *If* $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, *then find the matrix A.* **[CBSE 2013]**

SOLUTION Clearly, we have

$$A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}.$$

SCALAR MULTIPLICATION *Let A be a given matrix and k be a number. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k, to be denoted by kA.*

If A is an $(m \times n)$ matrix then kA is also an $(m \times n)$ matrix.

If $A = [a_{ij}]_{m \times n}$ then $kA = [k \cdot a_{ij}]_{m \times n}$.

EXAMPLE 7 *If* $A = \begin{bmatrix} 5 & 4 & -2 \\ 6 & -1 & 7 \end{bmatrix}$, *find (i)* $3A$ *(ii)* $\frac{1}{2}A$ *(iii)* $-2A$.

SOLUTION We have:

$$\text{(i)}\ \ 3A = \begin{bmatrix} 3 \cdot 5 & 3 \cdot 4 & 3 \cdot (-2) \\ 3 \cdot 6 & 3 \cdot (-1) & 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 & -6 \\ 18 & -3 & 21 \end{bmatrix}.$$

$$\text{(ii)}\ \ \frac{1}{2}A = \begin{bmatrix} \frac{1}{2} \cdot 5 & \frac{1}{2} \cdot 4 & \frac{1}{2} \cdot (-2) \\ \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot (-1) & \frac{1}{2} \cdot 7 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 2 & -1 \\ 3 & -\frac{1}{2} & \frac{7}{2} \end{bmatrix}.$$

$$\text{(iii)}\ \ -2A = (-2) \cdot A = \begin{bmatrix} (-2) \cdot 5 & (-2) \cdot 4 & (-2) \cdot (-2) \\ (-2) \cdot 6 & (-2) \cdot (-1) & (-2) \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -8 & 4 \\ -12 & 2 & -14 \end{bmatrix}.$$

Some Properties of Scalar Multiplication

THEOREM 1 *If A and B are two matrices of the same order and k is a scalar then prove that* $k(A + B) = kA + kB$.

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

$$\begin{aligned} k(A + B) &= k \cdot ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) \\ &= k \cdot [a_{ij} + b_{ij}]_{m \times n} && \text{[by definition of addition of matrices]} \\ &= [k(a_{ij} + b_{ij})]_{m \times n} && \text{[by definition of scalar multiplication]} \\ &= [k \cdot a_{ij} + k \cdot b_{ij}]_{m \times n} && \text{[by distributive law]} \\ &= [k \cdot a_{ij}]_{m \times n} + [k \cdot b_{ij}]_{m \times n} \\ &= kA + kB. \end{aligned}$$

Hence, $k(A + B) = kA + kB$.

THEOREM 2 *If A is any matrix and k_1, k_2 are any scalars then prove that*
(i) $(k_1 + k_2)A = k_1A + k_2A$ *(ii)* $k_1(k_2A) = (k_1k_2)A$.

PROOF Let $A = [a_{ij}]_{m\times n}$. Then,

(i) $(k_1 + k_2)A = (k_1 + k_2)\cdot[a_{ij}]_{m\times n}$

$= [(k_1 + k_2)\cdot a_{ij}]_{m\times n}$ [by definition of scalar multiplication]

$= [k_1 \cdot a_{ij} + k_2 \cdot a_{ij}]_{m\times n}$ [by distributive law]

$= [k_1 \cdot a_{ij}]_{m\times n} + [k_2 \cdot a_{ij}]_{m\times n}$

[by definition of addition of matrices]

$= k_1A + k_2A.$

Hence, $(k_1 + k_2)A = k_1A + k_2A$.

(ii) $k_1(k_2A) = k_1[k_2 \cdot a_{ij}]_{m\times n} = [k_1(k_2 \cdot a_{ij})]_{m\times n}$

$= [(k_1k_2)\cdot a_{ij}]_{m\times n}$

[by associativity of multiplication in numbers]

$= (k_1k_2)\cdot[a_{ij}]_{m\times n} = (k_1k_2)\ A.$

Hence, $k_1(k_2A) = (k_1k_2)A$.

SUMMARY

I. If A and B are comparable matrices and k is a scalar then
$k(A + B) = (kA + kB)$.

II. If k_1, k_2 are scalars and A is any matrix then
(i) $(k_1 + k_2)A = (k_1A + k_2A)$
(ii) $k_1(k_2A) = (k_1k_2)A$

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} 3 & 5 \\ 7 & -9 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & -4 \\ 2 & 3 \end{bmatrix}$, *find* $(4A - 3B)$.

SOLUTION $(4A - 3B) = 4A + (-3B)$.

Now, $4A = \begin{bmatrix} 4\cdot 3 & 4\cdot 5 \\ 4\cdot 7 & 4\cdot(-9) \end{bmatrix} = \begin{bmatrix} 12 & 20 \\ 28 & -36 \end{bmatrix}$.

And, $-3B = (-3)\cdot B = \begin{bmatrix} (-3)\cdot 6 & (-3)\cdot(-4) \\ (-3)\cdot 2 & (-3)\cdot 3 \end{bmatrix} = \begin{bmatrix} -18 & 12 \\ -6 & -9 \end{bmatrix}$.

$\therefore \quad 4A - 3B = 4A + (-3B)$

$= \begin{bmatrix} 12 & 20 \\ 28 & -36 \end{bmatrix} + \begin{bmatrix} -18 & 12 \\ -6 & -9 \end{bmatrix}$

$= \begin{bmatrix} 12 + (-18) & 20 + 12 \\ 28 + (-6) & -36 + (-9) \end{bmatrix} = \begin{bmatrix} -6 & 32 \\ 22 & -45 \end{bmatrix}$.

Hence, $(4A - 3B) = \begin{bmatrix} -6 & 32 \\ 22 & -45 \end{bmatrix}$.

EXAMPLE 2 *Let* $A = \text{diag}\,[3,\ -5,\ 7]$ *and* $B = \text{diag}\,[-1,\ 2,\ 4]$.
Find (i) $(A + B)$ *(ii)* $(A - B)$ *(iii)* $-5A$ *(iv)* $(2A + 3B)$.

SOLUTION We have

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$\therefore \text{ (i) } A + B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 11 \end{bmatrix}.$$

(ii) $(A - B) = A + (-B)$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\text{(iii) } -5A = (-5) \cdot A = (-5) \cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -35 \end{bmatrix}.$$

$$\text{(iv) } 2A + 3B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 26 \end{bmatrix}.$$

EXAMPLE 3 *Simplify* $\cos 0 \cdot \begin{bmatrix} \cos 0 & \sin 0 \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin 0 \cdot \begin{bmatrix} \sin 0 & -\cos 0 \\ \cos\theta & \sin\theta \end{bmatrix}$.

[CBSE 2012]

SOLUTION We have

$$\cos 0 \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin 0 & \cos 0 \end{bmatrix} + \sin 0 \cdot \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos 0 & \sin 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 0 & \sin\theta\ \cos\theta \\ -\sin 0\ \cos 0 & \cos s2\theta \end{bmatrix} + \begin{bmatrix} \sin^2 0 & -\sin\theta\ \cos\theta \\ \sin 0\ \cos 0 & \sin^2 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 0 + \sin^2 0 & \sin\theta\ \cos\theta + (-\sin\theta\ \cos\theta) \\ -\sin\theta\ \cos\theta + \sin\theta\ \cos\theta & \cos^2\theta + \sin^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

EXAMPLE 4 *If* $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix}+\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, *find* $(x-y)$. **[CBSE 2014]**

SOLUTION We have

$$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix}+\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix}+\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix}=\begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$\Rightarrow \quad 2x+1=5$ and $8+y=0$

[comparing the corresponding elements]

$\Rightarrow \quad x=2$ and $y=-8.$

$\Rightarrow \quad (x-y)=2-(-8)=10.$

EXAMPLE 5 *Find a matrix* X, *if* $X+\begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}=\begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

SOLUTION Let $A=\begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

Then, the given matrix equation is $X+A=B.$

$$\therefore \quad X+A=B \Rightarrow X=B+(-A)$$

$$=\begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}+\begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix}$$

$$=\begin{bmatrix} 3+(-4) & -6+(-6) \\ 5+3 & -8+(-7) \end{bmatrix}=\begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}.$$

Hence, $X=\begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$.

EXAMPLE 6 *Find a matrix* X *such that* $2A+B+X=O$, *where* $A=\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B=\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$. **[CBSE 2002C]**

SOLUTION We have

$$2A+B+X=O \Rightarrow X=-(2A+B).$$

$$\text{Now, } (2A+B)=2\cdot\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}+\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix}+\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 4+(-2) \\ 6+1 & 8+5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix}.$$

$$\therefore \quad X = -(2A + B) = \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}.$$

EXAMPLE 7 *Find matrices X and Y, if* $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ *and* $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

SOLUTION Adding the given matrices, we get

$$(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 5+3 & 2+6 \\ 0+0 & 9+(-1) \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow X = \frac{1}{2} \cdot \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}.$$

On subtracting the given matrices, we get

$$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 5-3 & 2-6 \\ 0-0 & 9-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

$$\Rightarrow \quad Y = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}.$$

Hence, $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$.

EXERCISE 5B

1. If $A = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & -2 \\ 4 & -3 & 1 \end{bmatrix}$, verify that $(A + B) = (B + A)$.

2. If $A = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 6 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 \\ 3 & -4 \\ 1 & 6 \end{bmatrix}$,
verify that $(A + B) + C = A + (B + C)$.

3. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 4 \\ 5 & -3 & 2 \end{bmatrix}$, find $(2A - B)$.

4. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$. Find:
(i) $A + 2B$ (ii) $B - 4C$ (iii) $A - 2B + 3C$

5. Let $A=\begin{bmatrix} 0 & 1 & -2 \\ 5 & -1 & -4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & 5 \end{bmatrix}$ and $C=\begin{bmatrix} 2 & -5 & 1 \\ -4 & 0 & 6 \end{bmatrix}$.
Compute $5A-3B+4C$.

6. If $5A=\begin{bmatrix} 5 & 10 & -15 \\ 2 & 3 & 4 \\ 1 & 0 & -5 \end{bmatrix}$, find A.

7. Find matrices A and B, if
$$A+B=\begin{bmatrix} 1 & 0 & 2 \\ 5 & 4 & -6 \\ 7 & 3 & 8 \end{bmatrix} \text{ and } A-B=\begin{bmatrix} -5 & -4 & 8 \\ 11 & 2 & 0 \\ -1 & 7 & 4 \end{bmatrix}.$$

8. Find matrices A and B, if
$$2A-B=\begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} \text{ and } 2B+A=\begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}.$$

9. Find matrix X, if $\begin{bmatrix} 3 & 5 & -9 \\ -1 & 4 & -7 \end{bmatrix}+X=\begin{bmatrix} 6 & 2 & 3 \\ 4 & 8 & 6 \end{bmatrix}$.

10. If $A=\begin{bmatrix} -2 & 3 \\ 4 & 5 \\ 1 & -6 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 2 \\ -7 & 3 \\ 6 & 4 \end{bmatrix}$, find a matrix C such that $A+B-C=O$.

11. Find the matrix X such that $2A-B+X=O$,
where $A=\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$.

12. If $A=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find a matrix C such that $(A+B+C)$ is a zero matrix.

13. If $A=\text{diag}\,[2,\ -5,\ 9]$, $B=\text{diag}\,[-3,\ 7,\ 14]$ and $C=\text{diag}\,[4,\ -6,\ 3]$, find:
(i) $A+2B$ (ii) $B+C-A$ (iii) $2A+B-5C$.

14. Find the values of x and y, when
(i) $\begin{bmatrix} x+y \\ x-y \end{bmatrix}=\begin{bmatrix} 8 \\ 4 \end{bmatrix}$
(ii) $\begin{bmatrix} 2x+5 & 7 \\ 0 & 3y-7 \end{bmatrix}=\begin{bmatrix} x-3 & 7 \\ 0 & -5 \end{bmatrix}$
(iii) $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix}+\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$ **[CBSE 2012]**

15. Find the value of $(x + y)$ from the following equation:

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$ **[CBSE 2012, '13C]**

16. If $\begin{bmatrix} x-y & 2y \\ 2y+z & x+y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$ then write the value of $(x + y + z)$. **[CBSE 2013C]**

ANSWERS (EXERCISE 5B)

3. $\begin{bmatrix} 8 & 2 & 0 \\ -3 & 7 & -8 \end{bmatrix}$ **4.** (i) $\begin{bmatrix} 4 & 10 \\ -1 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 9 & -17 \\ -14 & -11 \end{bmatrix}$ (iii) $\begin{bmatrix} -6 & 13 \\ 16 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 5 & -6 & -3 \\ 9 & 1 & -11 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 2 & -3 \\ \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ \frac{1}{5} & 0 & -1 \end{bmatrix}$

7. $A = \begin{bmatrix} -2 & -2 & 5 \\ 8 & 3 & -3 \\ 3 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & -3 \\ -3 & 1 & -3 \\ 4 & -2 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 & 4 \\ 0 & 0 & -6 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -3 & 12 \\ 5 & 4 & 13 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & 5 \\ -3 & 8 \\ 7 & -2 \end{bmatrix}$

11. $\begin{bmatrix} -8 & -1 \\ 0 & -1 \end{bmatrix}$ **12.** $\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$

13. (i) diag [–4, 11, 37] (ii) diag [–1, 6, 8] (iii) diag [–19, 27, 17]

14. (i) $x = 6,\ y = 2$ (ii) $x = -8,\ y = \frac{2}{3}$ (iii) $x = 2,\ y = 4$

15. $x + y = 6$ **16.** 10

Multiplication of Matrices

For two given matrices A and B, we say that the product AB exists only when the number of rows in A equals the number of columns in B.

When AB exists, we say that A is conformable to B for multiplication.

PRODUCT OF MATRICES

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A equals the number of rows in B.

Then, AB exists and it is an $(m \times p)$ matrix, given by

$$AB = [c_{ik}]_{m \times p}, \text{ where } c_{ik} = (a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots a_{in}b_{nk}) = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

$\therefore$ (i, k)th element of AB

= sum of the products of corresponding elements of ith row of A and kth column of B.

SUMMARY

(i) If A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix then AB exists and it is an $(m \times p)$ matrix.

(ii) (i, k)th element of AB
= sum of the products of corresponding elements of ith row of A and kth column of B.

REMARKS For two given matrices A and B:
(i) AB may exist and BA may not exist;
(ii) BA may exist and AB may not exist;
(iii) AB and BA both may not exist;
(iv) AB and BA both may exist.

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ *and* $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$ *then show that* AB *and* BA *both exist. Find* AB *and* BA.

SOLUTION Here, A is a 2×3 matrix and B is a 3×2 matrix.

$\therefore$ AB exists and it is a 2×2 matrix.

Let $AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$. Then,

$c_{11} = (\text{1st row of } A) \times (\text{1st column of } B)$
$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31};$

$c_{12} = (\text{1st row of } A) \times (\text{2nd column of } B)$
$= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32};$

$c_{21} = (\text{2nd row of } A) \times (\text{1st column of } B)$
$= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31};$

and $c_{22} = (\text{2nd row of } A) \times (\text{2nd column of } B)$

$$= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}.$$

$$\therefore \quad AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.$$

Again, B is a 3×2 matrix and A is a 2×3 matrix. So, BA exists and it is a 3×3 matrix.

Proceeding as above, we get

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}\,a_{11} + b_{12}\,a_{21} & b_{11}\,a_{12} + b_{12}\,a_{22} & b_{11}\,a_{13} + b_{12}\,a_{23} \\ b_{21}\,a_{11} + b_{22}\,a_{21} & b_{21}\,a_{12} + b_{22}\,a_{22} & b_{21}\,a_{13} + b_{22}\,a_{23} \\ b_{31}\,a_{11} + b_{32}\,a_{21} & b_{31}\,a_{12} + b_{32}\,a_{22} & b_{31}\,a_{13} + b_{32}\,a_{23} \end{bmatrix}.$$

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$, *find AB. Does BA exist?*

SOLUTION Here A is a 3×2 matrix and B is a 2×2 matrix.

Clearly, the number of columns in A equals the number of rows in B.

$\therefore$ AB exists and it is a 3×2 matrix.

Now, $AB = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot(-1) + (-1)\cdot 2 & 2\cdot 3 + (-1)\cdot 1 \\ 3\cdot(-1) + 4\cdot 2 & 3\cdot 3 + 4\cdot 1 \\ 1\cdot(-1) + 5\cdot 2 & 1\cdot 3 + 5\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 5 \\ 5 & 13 \\ 9 & 8 \end{bmatrix}.$$

Further, B is a 2×2 matrix and A is a 3×2 matrix. So, the number of columns in B is not equal to the number of rows in A.

So, BA does not exist.

EXAMPLE 3 *Let* $A=\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ *and* $B=\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}$. *Find AB and BA, and show that* $AB \neq BA$.

SOLUTION Here A is a 2×3 matrix and B is a 3×2 matrix.

So, AB exists and it is a 2×2 matrix.

$$\text{Now, } AB=\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot 2+(-2)\cdot 4+3\cdot(-2) & 1\cdot 3+(-2)\cdot 5+3\cdot 1 \\ (-4)\cdot 2+2\cdot 4+5\cdot(-2) & (-4)\cdot 3+2\cdot 5+5\cdot 1 \end{bmatrix}$$

$$=\begin{bmatrix} -12 & -4 \\ -10 & 3 \end{bmatrix}.$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, BA exists and it is a 3×3 matrix.

$$\text{Now, } BA=\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} 2\cdot 1+3\cdot(-4) & 2\cdot(-2)+3\cdot 2 & 2\cdot 3+3\cdot 5 \\ 4\cdot 1+5\cdot(-4) & 4\cdot(-2)+5\cdot 2 & 4\cdot 3+5\cdot 5 \\ (-2)\cdot 1+1\cdot(-4) & (-2)\cdot(-2)+1\cdot 2 & (-2)\cdot 3+1\cdot 5 \end{bmatrix}$$

$$=\begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -6 & 6 & -1 \end{bmatrix}.$$

Hence, $AB \neq BA$.

EXAMPLE 4 *If* $[2x \quad 4]\begin{bmatrix} x \\ -8 \end{bmatrix}=O$, *find the positive value of x.* **[CBSE 2014C]**

SOLUTION The given matrix equation is $AB=O$, where A is a (1×2) matrix and B is a (2×1) matrix. So, AB is a (1×1) matrix.

So, O is a (1×1) matrix.

$$\therefore \quad [2x \quad 4]\begin{bmatrix} x \\ -8 \end{bmatrix}=[0]$$

$$\Rightarrow \quad 2x^2-32=0 \Rightarrow \quad 2x^2=32$$

$$\Rightarrow \quad x^2=16 \Rightarrow \quad x=\sqrt{16}=4.$$

Hence, $x=4$.

Properties of Matrix Multiplication

1. ***Commutativity***
Matrix multiplication is not commutative in general.

PROOF Let A and B be two given matrices.

If AB exists then it is quite possible that BA may not exist.
For example, if A is a 3×2 matrix and B is a 2×2 matrix then clearly, AB exists but BA does not exist.

Similarly, if BA exists then AB may not exist.

For example, if A is a 2×3 matrix and B is a 2×2 matrix then clearly, BA exists but AB does not exist.

Further, if AB and BA both exist, they may not be comparable. For example, if A is a 2×3 matrix and B is a 3×2 matrix then clearly, AB as well as BA exists. But, AB is a 2×2 matrix while BA is a 3×3 matrix.

Again, if AB and BA both exist and they are comparable, even then they may not be equal.

For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then AB and BA are both defined and each one is a 2×2 matrix.

But, $AB = \begin{bmatrix} 1 & 5 \\ 1 & 8 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}$.

This shows that $AB \neq BA$.

Hence, in general, $AB \neq BA$.

REMARKS (i) When $AB = BA$, we say that A and B commute.
(ii) When $AB = -BA$, we say that A and B anticommute.

2. ***Associative law***
For any matrices A, B, C for which $(AB)C$ and $A(BC)$ *both exist, we have*
$(AB)C = A(BC)$.

3. ***Distributive laws of multiplication over addition***
We have:
(i) $A \cdot (B + C) = (AB + AC)$ *(ii)* $(A + B) \cdot C = (AC + BC)$

4. ***The product of two nonzero matrices can be a zero matrix.***

Example Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Then, $A \neq O$ and $B \neq O$.

And, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 2 + 1 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 2 + 2 \cdot 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Left zero divisor *If $AB = O$ and $A \neq O$ then A is called a left zero divisor of AB.*

Right zero divisor *If $AB = O$ and $B \neq O$ then B is called a right zero divisor of AB.*

5. ***If A is a given square matrix and I is an identity matrix of the same order as A then we have $A \cdot I = I \cdot A = A$.***
6. ***If A is a given square matrix and O is the null matrix of the same order as A then $O \cdot A = A \cdot O = O$.***

Positive Integral Powers of a Square Matrix

Let A be a square matrix of order n. Then, we define:

$A^2 = A \cdot A$;

$A^3 = A \cdot A \cdot A = A^2 \cdot A$;

$A^4 = A \cdot A \cdot A \cdot A = A^3 \cdot A$, and so on.

$\therefore \quad A^n = (A \cdot A \cdot A \cdot \ldots \cdot n \text{ times})$.

THEOREM 1 *If A and B are square matrices of the same order then*

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Also, when $AB = BA$ then $(A + B)^2 = A^2 + 2AB + B^2$.

PROOF Let A and B be n-rowed square matrices.
Then, clearly, $(A + B)$ is a square matrix of order n.
So, $(A + B)^2$ is defined.

Now, $(A + B)^2 = (A + B) \cdot (A + B)$

$= A \cdot (A + B) + B \cdot (A + B)$ [by distributive law]

$= AA + AB + BA + BB$ [by distributive law]

$= A^2 + AB + BA + B^2$.

Hence, $(A + B)^2 = (A^2 + AB + BA + B^2)$.

Particular case *When $AB = BA$*

In this case, we have

$(A + B)^2 = (A^2 + AB + AB + B^2) = (A^2 + 2AB + B^2)$ [$\because \quad BA = AB$].

THEOREM 2 *If A and B are square matrices of the same order then*

$$(A + B)(A - B) = A^2 - AB + BA - B^2.$$

Also, when $AB = BA$ then $(A + B)(A - B) = A^2 - B^2$.

PROOF We have:

$(A + B) \cdot (A - B) = A(A - B) + B(A - B)$ [by distributive law]

$= AA - AB + BA - BB$ [$\because \quad A(B - C) = AB - AC$]

$= A^2 - AB + BA - B^2$.

Hence, $(A + B)(A - B) = A^2 - AB + BA - B^2$.

Particular case *When* $AB = BA$
In this case, $(A + B)(A - B) = (A^2 - B^2)$ $[\because \quad BA = AB]$.

MATRIX POLYNOMIAL Let $f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_{m-1}x + a_m$ be a polynomial of degree m and let A be a square matrix of order n. Then, we define

$$f(A) = a_0A^m + a_1A^{m-1} + a_2A^{m-2} + \ldots + a_{m-1}A + a_mI,$$

where I is a unit matrix of order n.

SOLVED EXAMPLES

EXAMPLE 1 *If* $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$, *find AB and BA whichever exists.*

SOLUTION Here, A is a 2×2 matrix and B is a 2×3 matrix.
Clearly, the number of columns in A = number of rows in B.
$\therefore$ AB exists and it is a 2×3 matrix.

$$AB = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 3 + 4\cdot 6 & 5\cdot 5 + 4\cdot 8 & 5\cdot 1 + 4\cdot 4 \\ 2\cdot 3 + 3\cdot 6 & 2\cdot 5 + 3\cdot 8 & 2\cdot 1 + 3\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 15 + 24 & 25 + 32 & 5 + 16 \\ 6 + 18 & 10 + 24 & 2 + 12 \end{bmatrix} = \begin{bmatrix} 39 & 57 & 21 \\ 24 & 34 & 14 \end{bmatrix}.$$

Again, B is a 2×3 matrix and A is a 2×2 matrix.
$\therefore$ number of columns in $B \neq$ number of rows in A.
So, BA does not exist.

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$ *then find AB and BA.*

Show that $AB \neq BA$.

SOLUTION Here, A is a 2×3 matrix and B is a 3×2 matrix
So, number of columns in A = number of rows in B.
$\therefore$ AB exists and it is a 2×2 matrix.

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2 + (-1)\cdot 4 + 3\cdot 1 & 2\cdot 3 + (-1)\cdot(-2) + 3\cdot 5 \\ -4\cdot 2 + 5\cdot 4 + 1\cdot 1 & -4\cdot 3 + 5\cdot(-2) + 1\cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4+3 & 6+2+15 \\ -8+20+1 & -12-10+5 \end{bmatrix} = \begin{bmatrix} 3 & 23 \\ 13 & -17 \end{bmatrix}.$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, number of columns in B = number of rows in A.

$\therefore$ BA exists and it is a 3×3 matrix.

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2 + 3\cdot(-4) & 2\cdot(-1) + 3\cdot 5 & 2\cdot 3 + 3\cdot 1 \\ 4\cdot 2 + (-2)\cdot(-4) & 4\cdot(-1) + (-2)\cdot 5 & 4\cdot 3 + (-2)\cdot 1 \\ 1\cdot 2 + 5\cdot(-4) & 1\cdot(-1) + 5\cdot 5 & 1\cdot 3 + 5\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12 & -2+15 & 6+3 \\ 8+8 & -4-10 & 12-2 \\ 2-20 & -1+25 & 3+5 \end{bmatrix} = \begin{bmatrix} -8 & 13 & 9 \\ 16 & -14 & 10 \\ -18 & 24 & 8 \end{bmatrix}.$$

Clearly, $AB \neq BA$.

EXAMPLE 3 *If* $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$ *then verify that* $(AB)C = A(BC)$.

SOLUTION We have

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0-4 & 1-2+10 \\ 9+0-0 & 3+4+0 \\ -6+0-2 & -2+0+5 \end{bmatrix} = \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix}$$

$$\Rightarrow \quad (AB)C = \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+27 & -1+0 & 3-9 \\ 18+21 & 9+0 & -27-7 \\ -16+9 & -8+0 & 24-3 \end{bmatrix} = \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.$$

Also, $BC = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 6+3 & 3+0 & -9-1 \\ 0+6 & 0+0 & 0-2 \\ -4+15 & -2+0 & 6-5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A(BC) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-6+22 & 3-0-4 & -10+2+2 \\ 27+12+0 & 9+0-0 & -30-4+0 \\ -18+0+11 & -6+0-2 & 20-0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.$$

Hence, $(AB)C = A(BC)$.

EXAMPLE 4 *If* $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$, *verify that* $A(B + C) = (AB + AC)$.

SOLUTION We have

$$A(B+C) = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \left\{ \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 9 + 2\cdot 2 & 3\cdot(-1) + 2\cdot 2 & 3\cdot(-1) + 2\cdot 3 \\ 1\cdot 9 + 0\cdot 2 & 1\cdot(-1) + 0\cdot 2 & 1\cdot(-1) + 0\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 1 & 3 \\ 9 & -1 & -1 \end{bmatrix}.$$

Now, $AB = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot 1 + 2\cdot 0 & 3\cdot(-2) + 2\cdot 7 & 3\cdot 5 + 2\cdot 3 \\ 1\cdot 1 + 0\cdot 0 & 1\cdot(-2) + 0\cdot 7 & 1\cdot 5 + 0\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 21 \\ 1 & -2 & 5 \end{bmatrix}.$$

And, $AC = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot 8 + 2\cdot 2 & 3\cdot 1 + 2\cdot(-5) & 3\cdot(-6) + 2\cdot 0 \\ 1\cdot 8 + 0\cdot 2 & 1\cdot 1 + 0\cdot(-5) & 1\cdot(-6) + 0\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -7 & -18 \\ 8 & 1 & -6 \end{bmatrix}.$$

$$\therefore \quad (AB + AC) = \begin{bmatrix} 3 & 8 & 21 \\ 1 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 28 & -7 & -18 \\ 8 & 1 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 1 & 3 \\ 9 & -1 & -1 \end{bmatrix}.$$

Hence, $A(B + C) = AB + AC$.

EXAMPLE 5 *Give an example of two matrices A and B such that $A \neq O$, $B \neq O$ and $AB = BA = O$.*

SOLUTION Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $A \neq O$, $B \neq O$.

But, $AB = BA = O$.

EXAMPLE 6 *If* $A = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix}$ *and* $B = \begin{bmatrix} \cos^2\phi & \cos\phi \sin\phi \\ \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}$, *show that AB is a zero matrix if θ and ϕ differ by an odd multiple of* $\frac{\pi}{2}$.

SOLUTION We have

$$AB = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi \sin\phi \\ \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta \cos^2\phi + \cos\theta \sin\theta \cos\phi \sin\phi & \cos^2\theta \cos\phi \sin\phi + \cos\theta \sin\theta \sin^2\phi \\ \cos\theta \sin\theta \cos^2\phi + \sin^2\theta \cos\phi \sin\phi & \cos\theta \sin\theta \cos\phi \sin\phi + \sin^2\theta \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos 0 \ \cos\phi \cdot \cos(0-\phi) & \cos 0 \ \sin\phi \cdot \cos(0-\phi) \\ \sin 0 \ \cos\phi \cdot \cos(0-\phi) & \sin 0 \ \sin\phi \cdot \cos(0-\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \because (\theta-\phi) \text{ being an odd multiple of } (\pi/2), \\ \text{we have } \cos(\theta-\phi) = 0 \end{bmatrix}.$$

EXAMPLE 7 *If* $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, *show that* $A^2 - 5A - 14I = O$. **[CBSE 2004]**

SOLUTION We have

$$A^2 = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3 + (-5)(-4) & 3\cdot(-5) + (-5)\cdot 2 \\ -4\cdot 3 + 2\cdot(-4) & -4\cdot(-5) + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix};$$

$$-5A = (-5)\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix};$$

$$-14I = (-14)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}.$$

$$\therefore \quad A^2 - 5A - 14I = A^2 + (-5)A + (-14I)$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 + (-15) + (-14) & -25 + 25 + 0 \\ -20 + 20 + 0 & 24 + (-10) + (-14) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $A^2 - 5A - 14I = O$.

EXAMPLE 8 *If* $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, *find k so that* $A^2 = 8A + kI$. **[CBSE 2005C]**

SOLUTION We have

$$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix};$$

$$(8A + kI) = 8\cdot\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\therefore \quad A^2 = 8A + kI \ \Rightarrow \ \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\Rightarrow \ 8+k = 1 \text{ and } 56+k = 49 \ \Rightarrow \ k = -7.$$

Hence, $k = -7$.

EXAMPLE 9 *If* $f(x) = x^2 - 5x + 7$ *and* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, *find* $f(A)$. **[CBSE 2004C, '07C]**

SOLUTION We have $f(A) = A^2 - 5A + 7I$.

Now, $A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix};$$

$$-5A = \begin{bmatrix} (-5)\cdot 3 & (-5)\cdot 1 \\ (-5)\cdot(-1) & (-5)\cdot 2 \end{bmatrix} = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix};$$

$$7I = 7 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

$\therefore \quad f(A) = A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8+(-15)+7 & 5+(-5)+0 \\ -5+5+0 & 3+(-10)+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $f(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

EXAMPLE 10 *Find the matrix* A *such that* $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \cdot A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$.

SOLUTION Clearly, the product is a 3×3 matrix and the prefactor is a 3×2 matrix. So, A must be a 2×3 matrix.

Let $A = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$.

Then, the given equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

or $$\begin{bmatrix} 2x-u & 2y-v & 2z-w \\ x & y & z \\ -3x+4u & -3y+4v & -3z+4w \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}.$$

So, by the definition of equal matrices, we have

$$2x - u = -1;\ 2y - v = -8;\ 2z - w = -10;\ x = 1;\ y = -2;\ z = -5.$$

$\therefore \quad x = 1;\ y = -2;\ z = -5;\ u = 3;\ v = 4$ and $w = 0$.

Hence, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

EXAMPLE 11 *Find the value of x, if*

$$[1 \quad x \quad 1] \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O.$$ **[CBSE 2006C]**

SOLUTION We have

$$[1 \quad x \quad 1]_{1\times 3} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}_{3\times 1} = O$$

$$\Rightarrow \quad [1 + 2x + 15 \quad 3 + 5x + 3 \quad 2 + x + 2] \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \quad [16 + 2x \quad 6 + 5x \quad 4 + x] \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \quad [(16 + 2x) \cdot 1 + (6 + 5x) \cdot 2 + (4 + x) \cdot x] = O$$

$$\Rightarrow \quad (16 + 2x) + (12 + 10x) + (4x + x^2) = 0$$

$$\Rightarrow \quad x^2 + 16x + 28 = 0$$

$$\Rightarrow \quad (x + 14)\,(x + 2) = 0$$

$$\Rightarrow \quad x + 14 = 0 \text{ or } x + 2 = 0$$

$$\Rightarrow \quad x = -14 \quad \text{or} \quad x = -2.$$

Hence, $x = -14$ or $x = -2$.

EXAMPLE 12 *Solve for x and y, given that* $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. **[CBSE 2003C]**

SOLUTION We have

$$\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x + 2y \\ 3y + 2x \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} x + 2y = 3 & \text{... (i)} \\ 2x + 3y = 5 & \text{... (ii)} \end{cases}$$

Multiplying (i) by 2 and subtracting (ii) from it, we get $y = 1$.

Putting $y = 1$ in (i), we get $x = 1$.

Hence, $x = 1$ and $y = 1$.

EXAMPLE 13 *Let* $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ *and I is the identity matrix of order* 2.

Show that $(I + A) = (I - A) \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

SOLUTION Let $\tan\frac{\alpha}{2} = t$.

Then, $\cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} = \frac{1-t^2}{1+t^2}$

and $\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} = \frac{2t}{1+t^2}$.

$\therefore \quad (I + A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$.

And, $(I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$.

$\therefore \quad (I - A) \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1-t^2}{1+t^2} + \frac{2t^2}{1+t^2} & \frac{-2t}{1+t^2} + \frac{t(1-t^2)}{1+t^2} \\ \frac{-t(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} & \frac{2t^2}{1+t^2} + \frac{1-t^2}{1+t^2} \end{bmatrix}$

$= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} = (I + A)$.

Hence, $(I + A) = (I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

EXAMPLE 14 *If* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *then prove that*

$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in N$. **[CBSE 2004, '05, '06]**

SOLUTION We shall prove the result by using the principle of mathematical induction.

When $n = 1$, we have

$$A^1 = \begin{bmatrix} \cos 1\cdot\theta & \sin 1\cdot\theta \\ -\sin 1\cdot\theta & \cos 1\cdot\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Thus, the result is true for $n = 1$.

Let the result be true for $n = k$.

Then, $A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k0 & \cos k0 \end{bmatrix}$.

$\therefore \quad A^{k+1}$

$$= A\cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin 0 & \cos 0 \end{bmatrix}\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k0 & \cos k0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\ \cos k\theta - \sin\theta\ \sin k\theta & \cos\theta\ \sin k\theta + \sin\theta\ \cos k\theta \\ -\sin 0\ \cos k0 - \cos 0\ \sin k0 & -\sin 0\ \sin k0 + \cos 0\ \cos k0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, the result is true for $n = (k + 1)$, whenever it is true for $n = k$.

Hence, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ for all values of $n \in N$.

EXAMPLE 15 *If* $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ *then prove that* $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ *for all values of* $n \in N$.

SOLUTION We shall prove the result by using the principle of mathematical induction.

When $n = 1$, we have

$$A^1 = \begin{bmatrix} 1+2\cdot 1 & -4\cdot 1 \\ 1 & 1-2\cdot 1 \end{bmatrix} = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}.$$

Thus, the result is true for $n = 1$.

Let the result be true for $n = k$. Then, $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$.

$\therefore \quad A^{k+1} = A\cdot A^k$

$$= \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -12k-4+8k \\ 1+2k-k & -4k-1+2k \end{bmatrix} = \begin{bmatrix} 3+2k & -4k-4 \\ k+1 & 1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}.$$

Thus, the result is true for $n = (k+1)$, whenever it is true for $n = k$.

So, the result is true for all $n \in N$.

Hence, $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ for all values of $n \in N$.

EXAMPLE 16 *If* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, *prove that* $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ *for all values of* $n \in N$.

SOLUTION We shall prove the result by using the principle of mathematical induction.

When $n = 1$, we have

$$A^1 = \begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix} = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus, the result is true for $n = 1$.

Let it be true for $n = k$. Then, $A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$.

$\therefore \quad A^{k+1} = A \cdot A^k$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3(3^{k-1}) & 3(3^{k-1}) & 3(3^{k-1}) \\ 3(3^{k-1}) & 3(3^{k-1}) & 3(3^{k-1}) \\ 3(3^{k-1}) & 3(3^{k-1}) & 3(3^{k-1}) \end{bmatrix} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}.$$

Thus, the result is true for $n = (k+1)$, whenever it is true for $n = k$.

So, the result is true for all $n \in N$.

Hence, $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ for all values of $n \in N$.

EXAMPLE 17 *If* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *prove that for all* $n \in N$,

$$(aI + bA)^n = a^nI + na^{n-1}bA,$$

where I is the identity matrix of order 2.

SOLUTION We shall prove the result by mathematical induction.

When $n = 1$, we have:

$$\text{LHS} = (aI + bA)^1 = (aI + bA) = (a^1I + 1a^0bA) = \text{RHS}.$$

So, the result is true for $n = 1$.

Let it be true for $n = m$, so that

$$(aI + bA)^m = a^mI + ma^{m-1}bA \qquad \ldots \text{(i)}$$

$$\therefore \quad (aI + bA)^{m+1} = (aI + bA) \cdot (aI + bA)^m$$

$$= (aI + bA) \cdot (a^mI + ma^{m-1}bA) \qquad \text{[using (i)]}$$

$$= aI(a^mI + ma^{m-1}bA) + bA(a^mI + ma^{m-1}bA)$$

$$= a^{m+1}I + ma^mbA + a^mbA + ma^{m-1}b^2A^2$$

$$[\because \quad II = I, \ IA = A = AI]$$

$$= a^{m+1} + (m + 1)\, a^mbA \qquad [\because \quad A^2 = 0].$$

This shows that the result is true for $n = (m + 1)$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

EXAMPLE 18 *If* $A = \text{diag } [a, b, c]$, *show that* $A^n = \text{diag } [a^n, b^n, c^n]$ *for all* $n \in N$.

SOLUTION We shall prove the result by mathematical induction.

When $n = 1$, we have

$$A^1 = \text{diag } [a^1, \ b^1, \ c^1] = \text{diag } [a, \ b, \ c] = A.$$

So, the result is true for $n = 1$.

Let it be true for $n = m$, so that

$$A^m = \text{diag } [a^m, \ b^m, \ c^m] \qquad \ldots \text{(i)}$$

$$\therefore \quad A^{m+1} = A \cdot A^m$$

$$= \text{diag } [a, \ b, \ c] \cdot \text{diag } [a^m, \ b^m, \ c^m] \text{ [using (i)]}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \cdot \begin{bmatrix} a^m & 0 & 0 \\ 0 & b^m & 0 \\ 0 & 0 & c^m \end{bmatrix}$$

$$= \begin{bmatrix} a^{m+1} & 0 & 0 \\ 0 & b^{m+1} & 0 \\ 0 & 0 & c^{m+1} \end{bmatrix} = \text{diag } [a^{m+1}, \ b^{m+1}, \ c^{m+1}].$$

This shows that the result is true for $n = (m + 1)$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

EXAMPLE 19 *If A is an $m \times n$ matrix and B is a matrix given in such a way that AB and BA are both defined, show that B is an $n \times m$ matrix.*

SOLUTION Since AB is defined, we have

number of rows in B = number of columns in $A = n$.

Again, since BA is defined, we have

number of columns in B = number of rows in $A = m$.

Hence, the order of B is $n \times m$.

EXAMPLE 20 *If A and B are two matrices given in such a way that AB and $A + B$ are both defined, show that A and B are square matrices of the same order.*

SOLUTION Since $(A + B)$ is defined, it follows that both A and B are of the same order, say $(m \times n)$.

Thus, order of A is $m \times n$ and order of B is $m \times n$.

But, AB is defined.

So, number of columns in A = number of rows in B.

Consequently, $n = m$.

$\therefore$ A and B are square matrices of the same order.

VALUE BASED QUESTION

EXAMPLE 21 *The cooperative store of a particular school has* 10 *dozen physics books,* 8 *dozen chemistry books and* 5 *dozen mathematics books. Their selling prices are* ₹ 65.70, ₹ 43.20 *and* ₹ 76.50 *respectively. Find by matrix method the total amount received by the store from selling all these items.*

SOLUTION We have

$$[\text{Total cost}] = \underset{\text{Number of books}}{\begin{bmatrix} \text{physics} & \text{chemistry} & \text{mathematics} \\ 120 & 96 & 60 \end{bmatrix}} \underset{\text{Rates}}{\begin{bmatrix} ₹\,65.70 \\ ₹\,43.20 \\ ₹\,76.50 \end{bmatrix}}$$

$$= [₹\,(120 \times 65.70 + 96 \times 43.20 + 60 \times 76.50)]$$

$$= [₹\,16621.20].$$

$\therefore$ the total amount realised = ₹ 16621.20.

EXERCISE 5C

1. Compute AB and BA, whichever exists when

(i) $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$

(ii) $A=\begin{bmatrix} -1 & 1 \\ -2 & 2 \\ -3 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ -3 & 4 & -5 \end{bmatrix}$

(iii) $A=\begin{bmatrix} 0 & 1 & -5 \\ 2 & 4 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 0 & 5 \end{bmatrix}$

(iv) $A=[1 \;\; 2 \;\; 3 \;\; 4]$ and $B=\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(v) $A=\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

2. Show that $AB \neq BA$ in each of the following cases:

(i) $A=\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

(ii) $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

3. Show that $AB = BA$ in each of the following cases:

(i) $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B=\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

(ii) $A=\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$

(iii) $A=\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$

4. If $A=\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that $AB = A$ and $BA = B$.

5. If $A=\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B=\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, show that AB is a zero matrix.

6. For the following matrices, verify that $A(BC) = (AB)C$:

(i) $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $C = [1 \quad -2]$

7. Verify that $A(B + C) = (AB + AC)$, when

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.

8. If $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -4 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$;
verify that $A(B - C) = (AB - AC)$.

9. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, show that $A^2 = O$.

10. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that $A^2 = A$.

11. If $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, show that $A^2 = I$.

12. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find $(3A^2 - 2B + I)$. **[CBSE 2005]**

13. If $A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$ then find $(-A^2 + 6A)$.

14. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $(A^2 - 5A + 7I) = O$. **[CBSE 2003C]**

15. Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 4A^2 + A = O$. **[CBSE 2005]**

16. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k so that $A^2 = kA - 2I$. **[CBSE 2003]**

17. If $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 2x + 3$. **[CBSE 2004C]**

18. If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and $f(x) = 2x^3 + 4x + 5$, find $f(A)$.

19. Find the values of x and y, when
$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$ **[CBSE 2003C]**

20. Solve for x and y, when
$$\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}.$$ **[CBSE 2005C]**

21. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y such that $A^2 + xI = yA$. **[CBSE 2005]**

22. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the values of a and b such that $A^2 + aA + bI = O$.

[CBSE 2006]

23. Find the matrix A such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \cdot A = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

24. Find the matrix A such that $A \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$.

25. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & -1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = (A^2 + B^2)$ then find the values of a and b.

26. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x) \cdot F(y) = F(x + y)$.

27. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^2 = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$.

28. If $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = O$, find x. **[CBSE 2005]**

29. If $\begin{bmatrix} x & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ -1 \end{bmatrix} = O$, find x.

30. Find the values of a and b for which
$$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$ **[CBSE 2003C]**

31. If $A = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 5x + 7$. **[CBSE 2004C]**

32. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all $n \in N$.

33. Give an example of two matrices A and B such that
$$A \neq O,\ B \neq O,\ AB = O \text{ and } BA \neq O.$$

34. Give an example of three matrices A, B, C such that
$$AB = AC \text{ but } B \neq C.$$

35. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find $(3A^2 - 2B + I)$. **[CBSE 2005]**

36. If $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$, find the value of x. **[CBSE 2012]**

ANSWERS (EXERCISE 5C)

1. (i) $AB = \begin{bmatrix} -4 & 2 \\ -6 & 9 \\ 2 & 13 \end{bmatrix}$ and BA does not exist.

(ii) $BA = \begin{bmatrix} -2 & 2 \\ -8 & 8 \\ 10 & -10 \end{bmatrix}$ and AB does not exist.

(iii) $AB = \begin{bmatrix} -1 & -25 \\ -2 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 6 & 13 & -5 \\ 0 & -1 & 5 \\ 10 & 20 & 0 \end{bmatrix}$

(iv) $AB = [30]$ and $BA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$

(v) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 4 & -20 \\ 38 & -10 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **16.** $k = 1$ **17.** $\begin{bmatrix} 12 & -4 \\ -6 & 8 \end{bmatrix}$

18. $f(A)=\begin{bmatrix} -5 & 68 \\ 136 & -141 \end{bmatrix}$ **19.** $x=2,\ y=1$ **20.** $x=5,\ y=3$

21. $x=8,\ y=8$ **22.** $a=-4,\ b=1$ **23.** $A=\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ **24.** $A=\begin{bmatrix} -8 & 4 \\ -19 & 12 \end{bmatrix}$

25. $a=1,\ b=4.$ **28.** $x=\dfrac{-5}{3}$ **29.** $x=-2$ or $x=-1$

30. $a=1,\ b=-3$ **31.** $f(A)=\begin{bmatrix} -15 & -20 \\ 20 & 15 \end{bmatrix}$ **33.** $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

34. $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ C=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **35.** $\begin{bmatrix} 4 & -8 \\ -22 & 134 \end{bmatrix}$ **36.** $x=13$

Transpose of Matrices

TRANSPOSE OF A MATRIX *Let A be an $(m \times n)$ matrix. Then, the matrix obtained by interchanging the rows and columns of A is called the transpose of A, denoted by A' or A^T.*

Thus, if $A=[a_{ij}]_{m\times n}$ then $A'=[a_{ji}]_{n\times m}$.

REMARKS (i) If A is an $(m \times n)$ matrix then A' is an $(n \times m)$ matrix.
(ii) (i, j)th element of $A=(j, i)$th element of A'.

EXAMPLE If $A=\begin{bmatrix} 2 & \sqrt{2} & 0 \\ 3 & -2 & \dfrac{2}{5} \end{bmatrix}$, then $A'=\begin{bmatrix} 2 & 3 \\ \sqrt{2} & -2 \\ 0 & \dfrac{2}{5} \end{bmatrix}$.

Some Results on Transpose of Matrices

THEOREM 1 *For any matrix A, prove that $(A')'=A$.*

PROOF Let $A=[a_{ij}]_{m\times n}$. Then,

A is an $m \times n$ matrix $\Rightarrow$ A' is an $n \times m$ matrix

$\Rightarrow$ $(A')'$ is an $m \times n$ matrix.

$\therefore$ A and $(A')'$ are matrices of the same order.

Also, (i, j)th element of $A=(j, i)$th element of A'

$=(i, j)$th element of $(A')'$.

$\therefore$ (i, j)th element of $A=(i, j)$th element of $(A')'$.

Thus, A and $(A')'$ are comparable matrices having their corresponding elements equal.

Hence, $(A')'=A$.

THEOREM 2 *If A is any matrix and k is a scalar, prove that $(kA)' = kA'$.*

PROOF Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix and k be a scalar.

Then, A is an $m \times n$ matrix $\Rightarrow$ kA is an $m \times n$ matrix

$\Rightarrow$ $(kA)'$ is an $n \times m$ matrix.

Again, A is an $m \times n$ matrix $\Rightarrow$ A' is an $n \times m$ matrix

$\Rightarrow$ kA' is an $n \times m$ matrix.

Thus, $(kA)'$ and kA' are matrices of the same order.

Now, (j, i)th element of $(kA)' = (i, j)$th element of kA

$= k$ times (i, j)th element of A

$= k$ times (j, i)th element of A'

$= (j, i)$th element of kA'.

$\therefore$ (j, i)th element of $(kA)' = (j, i)$th element of kA'.

Thus, $(kA)'$ and kA' are comparable matrices having their corresponding elements equal.

Hence, $(kA)' = kA'$.

THEOREM 3 *If A and B are two matrices of the same order then prove that $(A + B)' = A' + B'$.*

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

A is an $m \times n$ matrix and B is an $m \times n$ matrix

$\Rightarrow$ $(A + B)$ is an $m \times n$ matrix

$\Rightarrow$ $(A + B)'$ is an $n \times m$ matrix.

Also, A is an $m \times n$ matrix and B is an $m \times n$ matrix

$\Rightarrow$ A' is an $n \times m$ matrix and B' is an $n \times m$ matrix

$\Rightarrow$ $(A' + B')$ is an $n \times m$ matrix.

$\therefore$ $(A + B)'$ and $(A' + B')$ are comparable matrices.

Also, (j, i)th element of $(A + B')$

$= (i, j)$th element of $(A + B)$

$= (i, j)$th element of $A + (i, j)$th element of B

$= (j, i)$th element of $A' + (j, i)$th element of B'

$= (j, i)$th element of $(A' + B')$.

$\therefore$ (j, i)th element of $(A + B') = (j, i)$th element of $(A' + B')$.

Thus, $(A + B)'$ and $(A' + B')$ are comparable and their corresponding elements are equal.

Hence, $(A + B)' = A' + B'$.

THEOREM 4 *If A and B are square matrices of the same order then $(AB)' = B'A'$.*

REMARK The proof of this theorem is beyond the scope of this book.

SUMMARY

(i) $(A')' = A$ (ii) $(kA)' = kA'$

(iii) $(A + B)' = A' + B'$ (iv) $(AB)' = B'A'$

SOLVED EXAMPLES

EXAMPLE 1 *Let* $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix}$. *Verify that* $(A')' = A$.

SOLUTION We have

$$A' = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 3 & -5 \\ -1 & 7 \end{bmatrix}$$

$$\Rightarrow \quad (A')' = \begin{bmatrix} 2 & 0 \\ 3 & -5 \\ -1 & 7 \end{bmatrix}' = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \end{bmatrix} = A.$$

Hence, $(A')' = A$.

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & 3 & -5 \\ 0 & -1 & 4 \end{bmatrix}$, *verify that* $(3A)' = 3A'$.

SOLUTION We have

$$3A = \begin{bmatrix} 3 \times 2 & 3 \times 3 & 3 \times (-5) \\ 3 \times 0 & 3 \times (-1) & 3 \times 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 & -15 \\ 0 & -3 & 12 \end{bmatrix}.$$

$$\therefore \quad (3A)' = \begin{bmatrix} 6 & 9 & -15 \\ 0 & -3 & 12 \end{bmatrix}' = \begin{bmatrix} 6 & 0 \\ 9 & -3 \\ -15 & 12 \end{bmatrix}.$$

$$\text{Also, } A' = \begin{bmatrix} 2 & 3 & -5 \\ 0 & -1 & 4 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ -5 & 4 \end{bmatrix}.$$

$$\therefore \quad 3A' = 3 \cdot \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 9 & -3 \\ -15 & 12 \end{bmatrix}.$$

Hence, $(3A)' = 3A'$.

EXAMPLE 3 *Let* $A = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix}$. *Verify that* $(A + B)' = A' + B'$.

SOLUTION We have

$$A + B = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2 & 4+(-3) \\ -2+5 & 0+6 \\ 7+(-1) & -5+8 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 6 \\ 6 & 3 \end{bmatrix}.$$

$$\therefore \quad (A + B)' = \begin{bmatrix} 5 & 1 \\ 3 & 6 \\ 6 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & 3 & 6 \\ 1 & 6 & 3 \end{bmatrix}.$$

$$\text{Also, } A' = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix}' = \begin{bmatrix} 3 & -2 & 7 \\ 4 & 0 & -5 \end{bmatrix}.$$

$$\text{And, } B' = \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix}' = \begin{bmatrix} 2 & 5 & -1 \\ -3 & 6 & 8 \end{bmatrix}.$$

$$\therefore \quad (A' + B') = \begin{bmatrix} 3 & -2 & 7 \\ 4 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -1 \\ -3 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2 & -2+5 & 7+(-1) \\ 4+(-3) & 0+6 & -5+8 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 6 \\ 1 & 6 & 3 \end{bmatrix}.$$

Hence, $(A + B)' = A' + B'$.

EXAMPLE 4 *If* $A = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ *and* $B = [1 \;\; 6 \;\; -4]$ *then verify that* $(AB)' = B'A'$.

[CBSE 2002]

SOLUTION We have $A = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ and $B = [1 \;\; 6 \;\; -4]$.

$$\therefore \quad AB = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} [1 \;\; 6 \;\; -4]$$

$$= \begin{bmatrix} -3 & -18 & 12 \\ 5 & 30 & -20 \\ 2 & 12 & -8 \end{bmatrix}.$$

$$\text{So, } (AB)' = \begin{bmatrix} -3 & 5 & 2 \\ -18 & 30 & 12 \\ 12 & -20 & -8 \end{bmatrix}.$$

$$\text{Also, } A' = [-3 \;\; 5 \;\; 2] \text{ and } B' = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}.$$

$$\therefore \quad B'A' = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix} [-3 \;\; 5 \;\; 2]$$

$$= \begin{bmatrix} -3 & 5 & 2 \\ -18 & 30 & 12 \\ 12 & -20 & -8 \end{bmatrix}.$$

Hence, $(AB)' = B'A'$.

SYMMETRIC AND SKEW-SYMMETRIC MATRICES

SYMMETRIC MATRIX *A square matrix A is said to be symmetric if* $A' = A$.

A is symmetric $\Leftrightarrow \; a_{ji} = a_{ij}$.

EXAMPLE 1 *Consider the matrix* $A = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & \frac{1}{3} \end{bmatrix}$.

$$\text{Then, } A' = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & \frac{1}{3} \end{bmatrix}' = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & \frac{1}{3} \end{bmatrix} = A.$$

Hence, A is symmetric.

EXAMPLE 2 *Consider the matrix* $B = \begin{bmatrix} 6 & -7 & 4 \\ -7 & 3 & 0 \\ 4 & 0 & \sqrt{5} \end{bmatrix}$.

$$\text{Then, } B' = \begin{bmatrix} 6 & -7 & 4 \\ -7 & 3 & 0 \\ 4 & 0 & \sqrt{5} \end{bmatrix}' = \begin{bmatrix} 6 & -7 & 4 \\ -7 & 3 & 0 \\ 4 & 0 & \sqrt{5} \end{bmatrix} = B.$$

Hence, B is symmetric.

SKEW-SYMMETRIC MATRIX *A square matrix A is said to be skew-symmetric if* $A' = -A$.

REMARK A is skew-symmetric $\Leftrightarrow \; A' = -A \Leftrightarrow a_{ji} = -a_{ij}$

$\Leftrightarrow \; a_{ii} = -a_{ii} \Leftrightarrow 2a_{ii} = 0 \Leftrightarrow a_{ii} = 0.$

Thus, every diagonal element of a skew-symmetric matrix is zero.

EXAMPLE 1 *Let* $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$. Then,

$$A' = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = -A.$$

Hence, A is skew-symmetric.

EXAMPLE 2 *Let* $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$. Then,

$$A' = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix} = -A.$$

Hence, A is skew-symmetric.

SOME RESULTS ON SYMMETRIC AND SKEW-SYMMETRIC MATRICES

THEOREM 1 *Prove that the sum of two symmetric matrices is a symmetric matrix.*

PROOF Let A and B be symmetric matrices of the same order.
Then, $A' = A$ and $B' = B$.

$\therefore\ (A + B)' = (A' + B') = (A + B)$ [$\because\ A' = A$ and $B' = B$].

Hence, $(A + B)$ is symmetric.

THEOREM 2 *If A is a symmetric matrix then prove that kA is symmetric.*

PROOF Since A is symmetric, we have $A' = A$.

$\therefore\ (kA)' = k \cdot A' = kA$ [$\because\ A' = A$].

This shows that kA is symmetric.

THEOREM 3 *Prove that the sum of two skew-symmetric matrices is a skew-symmetric matrix.*

PROOF Let A and B be two skew-symmetric matrices.
Then, $A' = -A$ and $B' = -B$.

$\therefore\ (A + B)' = (A' + B') = (-A) + (-B) = -(A + B)$.

Hence, $(A + B)$ is skew-symmetric.

THEOREM 4 *If A is a skew-symmetric matrix then prove that kA is skew-symmetric.*

PROOF Since A is skew-symmetric, we have $A' = -A$.

$\therefore\ (kA)' = k \cdot A' = k \cdot (-A)$ [$\because\ A' = -A$]

$= -(k \cdot A)$.

Thus, $(kA)' = -(kA)$.

Hence, (kA) is skew-symmetric.

THEOREM 5 *For any square matrix A with real number entries, prove that (i) $(A + A')$ is symmetric and (ii) $(A - A')$ is skew-symmetric.*

PROOF Let A be a square matrix with real number entries. Then, we have:

(i) $(A + A')' = A' + (A')' \quad [\because \ (A + B)' = A' + B']$

$= A' + A \quad [\because \ (A')' = A]$

$= (A + A') \quad [\because \ (A + B) = (B + A)].$

Thus, $(A + A')' = (A + A')$.

Hence, $(A + A')$ is symmetric.

(ii) $(A - A')' = A' - (A')' \quad [\because \ (A - B)' = A' - B']$

$= A' - A \quad [\because \ (A')' = A]$

$= -(A - A').$

Thus, $(A - A')' = -(A - A')$

Hence, $(A - A')$ is skew-symmetric.

THEOREM 6 *Prove that every square matrix is expressible as the sum of a symmetric and a skew-symmetric matrix.*

PROOF Let A be any square matrix. Then, we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q \text{ (say)},$$

where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$.

Now, $P' = \left\{\frac{1}{2}(A + A')\right\}'$

$= \frac{1}{2}(A + A')' \quad [\because \ (kA)' = kA']$

$= \frac{1}{2}\{A' + (A')'\} \quad [\because \ (A + B)' = (A' + B')]$

$= \frac{1}{2}(A' + A) \quad [\because \ (A')' = A]$

$= \frac{1}{2}(A + A') \quad [\because \ (A' + A) = (A + A')]$

$= P.$

Thus, $P' = P$ and therefore, P is symmetric.

And, $Q' = \left\{\frac{1}{2}(A - A')\right\}'$

$= \frac{1}{2}(A - A') \quad [\because \ (kA)' = kA']$

$= \frac{1}{2}\{A' - (A')'\} \quad [\because \ (A - B)' = A' - B']$

$= \frac{1}{2}(A' - A) \quad [\because \ (A')' = A]$

$= -\frac{1}{2}(A - A') = -Q.$

Thus, $Q' = -Q$ and therefore, Q is skew-symmetric.

$\therefore$ $A = P + Q$, where P is symmetric and Q is skew-symmetric.

SOLVED EXAMPLES

EXAMPLE 1 *Express the matrix* $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ *as the sum of a symmetric matrix and a skew-symmetric matrix.*

SOLUTION We know that $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.

Here, $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$.

$$\therefore \quad (A + A') = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & -4+1 \\ 1+(-4) & -1+(-1) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & -2 \end{bmatrix}.$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2} \cdot \begin{bmatrix} 6 & -3 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix}.$$

$$\text{And, } (A - A') = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & -4-1 \\ 1-(-4) & -1-(-1) \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1+4 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}.$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \frac{1}{2} \cdot \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}.$$

$$\text{Then, } P' = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix}' = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix} = P.$$

$\therefore$ P is symmetric.

$$\text{And, } Q' = \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}' = \begin{bmatrix} 0 & \frac{5}{2} \\ \frac{-5}{2} & 0 \end{bmatrix} = -Q.$$

$\therefore$ Q is skew-symmetric.

Now, $P+Q=\begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix}+\begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}=A.$

Hence, $A = P + Q$, where P is symmetric and Q is skew-symmetric.

EXAMPLE 2 *Express the matrix* $A=\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ *as the sum of a symmetric and a skew-symmetric matrix.* **[CBSE 2006]**

SOLUTION We have

$$A=\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} \Rightarrow A'=\begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}.$$

$$\therefore \quad (A+A')=\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}+\begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} 1+1 & 3+(-6) & 5+(-4) \\ -6+3 & 8+8 & 3+6 \\ -4+5 & 6+3 & 5+5 \end{bmatrix}=\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}.$$

$$\text{Let } P=\frac{1}{2}(A+A')=\frac{1}{2}\cdot\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}=\begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix}.$$

$$\text{And, } (A-A')=\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}-\begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} 1-1 & 3-(-6) & 5-(-4) \\ -6-3 & 8-8 & 3-6 \\ -4-5 & 6-3 & 5-5 \end{bmatrix}=\begin{bmatrix} 0 & 3+6 & 5+4 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}.$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \frac{1}{2}\cdot\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ \frac{-9}{2} & 0 & \frac{-3}{2} \\ \frac{-9}{2} & \frac{3}{2} & 0 \end{bmatrix}.$$

$$P' = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix}' = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} = P.$$

$\therefore$ P is symmetric.

$$\text{And, } Q' = \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ \frac{-9}{2} & 0 & \frac{-3}{2} \\ \frac{-9}{2} & \frac{3}{2} & 0 \end{bmatrix}' = \begin{bmatrix} 0 & \frac{-9}{2} & \frac{-9}{2} \\ \frac{9}{2} & 0 & \frac{3}{2} \\ \frac{9}{2} & \frac{-3}{2} & 0 \end{bmatrix} = -Q.$$

$\therefore$ Q is skew-symmetric.

$$\text{Now, } (P+Q) = \begin{bmatrix} 1 & \frac{-3}{2} & \frac{1}{2} \\ \frac{-3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ \frac{-9}{2} & 0 & \frac{-3}{2} \\ \frac{-9}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} = A.$$

Hence, $A = P + Q$,
where P is symmetric and Q is skew-symmetric.

EXAMPLE 3 *If A and B are symmetric matrices of the same order then show that AB is symmetric if and only if $AB = BA$.*

SOLUTION Let A and B be symmetric matrices.

Then, $A' = A$ and $B' = B$.

Let AB be symmetric. Then,

$AB = (AB)'$ [by definition]

$= B'A' = BA$ [$\because$ $B' = B$ and $A' = A$].

$\therefore$ $AB = BA$.

Conversely, let $AB = BA$. Then,

$(AB)' = B'A'$

$= BA$ [$\because$ $B' = B$ and $A' = A$]

$= AB$ [given].

Thus, $(AB)' = AB$ and hence, AB is symmetric.

EXAMPLE 4 *If A and B are symmetric matrices, prove that $(AB - BA)$ is skew-symmetric.*

SOLUTION Let A and B be symmetric matrices.

Then, $A' = A$ and $B' = B$.

Now, $(AB - BA)' = (AB)' - (BA)'$

$= (B'A') - (A'B')$

$= (BA - AB)$ [$\because$ $A' = A$ and $B' = B$]

$= -(AB - BA)$.

Thus, $(AB - BA)' = -(AB - BA)$.

Hence, $(AB - BA)$ is skew-symmetric.

EXAMPLE 5 *If A is symmetric, show that $B'AB$ is symmetric.*

SOLUTION Let A be symmetric. Then, $A' = A$.

$\therefore$ $(B'AB)' = B'A'(B')'$

$= B'A'B$ [$\because$ $(B')' = B$]

$= B'AB$ [$\because$ $A' = A$].

Thus, $(B'AB)' = B'AB$.

Hence, $(B'AB)$ is symmetric.

EXERCISE 5D

1. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 7 & -4 \end{bmatrix}$, verify that $(A')' = A$.

2. If $A = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 4 & -6 \end{bmatrix}$, verify that $(2A)' = 2A'$.

3. If $A = \begin{bmatrix} 3 & 2 & -1 \\ -5 & 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -5 & -2 \\ 3 & 1 & 8 \end{bmatrix}$, verify that $(A + B)' = (A' + B')$.

4. If $P = \begin{bmatrix} 3 & 4 \\ 2 & -1 \\ 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 7 & -5 \\ -4 & 0 \\ 2 & 6 \end{bmatrix}$, verify that $(P + Q)' = (P' + Q')$.

5. If $A = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix}$, show that $(A + A')$ is symmetric. **[CBSE 2001]**

6. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $(A - A')$ is skew-symmetric. **[CBSE 2001C]**

7. Show that the matrix $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is skew-symmetric.

HINT: Show that $A' = -A$.

8. Express the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

9. Express the matrix $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

10. Express the matrix $A = \begin{bmatrix} -1 & 5 & 1 \\ 2 & 3 & 4 \\ 7 & 0 & 9 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

11. Express the matrix A as the sum of a symmetric and a skew-symmetric matrix, where

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{bmatrix}.$$ **[CBSE 2005C]**

12. Express the matrix $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$ as sum of two matrices such that one is symmetric and the other is skew-symmetric. **[CBSE 2008]**

13. For each of the following pairs of matrices A and B, verify that $(AB)' = (B'A')$:

(i) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 \quad -1 \quad -4]$ **[CBSE 2002]**

(iv) $A = \begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}$

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A'A = I$.

15. If matrix $A = [1 \quad 2 \quad 3]$, write AA'. **[CBSE 2009]**

ANSWERS (EXERCISE 5D)

8. $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & \frac{-3}{2} \\ \frac{-3}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} -1 & \frac{7}{2} & 4 \\ \frac{7}{2} & 3 & 2 \\ 4 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & -3 \\ \frac{-3}{2} & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 3 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-3}{2} & \frac{-1}{2} \\ \frac{3}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 & 3 & \frac{5}{2} \\ 3 & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & \frac{-3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix}$

15. [14]

Elementary Operations on Matrices

Given below are three row operations and three column operations on a matrix, which are called elementary operations or transformations.

EQUIVALENT MATRICES *Two matrices A and B are said to be equivalent if one is obtained from the other by one or more elementary operations and we write, A ~ B.*

THREE ELEMENTARY ROW OPERATIONS

(i) Interchange of any two rows The interchange of *i*th and *j*th rows is denoted by $R_i \leftrightarrow R_j$.

EXAMPLE Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{2} & 4 & 6 \\ 5 & -3 & 7 \end{bmatrix}$.

$$\text{Applying } R_2 \leftrightarrow R_3, \text{ we get } A \sim \begin{bmatrix} 3 & 2 & -1 \\ 5 & -3 & 7 \\ \sqrt{2} & 4 & 6 \end{bmatrix}.$$

(ii) Multiplication of the elements of a row by a nonzero number Suppose each element of *i*th row of a given matrix is multiplied by a nonzero number *k*.

Then, we denote it by $\boldsymbol{R_i \to k\, R_i}$.

EXAMPLE Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{3} & -5 & 6 \\ 1 & 8 & 4 \end{bmatrix}$.

$$\text{Applying } R_2 \to 4R_2, \text{ we get } A \sim \begin{bmatrix} 3 & 2 & -1 \\ 4\sqrt{3} & -20 & 24 \\ 1 & 8 & 4 \end{bmatrix}.$$

(iii) Multiplying each element of a row by a nonzero number and then adding them to the corresponding elements of another row Suppose each element of *j*th row of a matrix A is multiplied by a nonzero number k and then added to the corresponding elements of *i*th row.

We denote it by $\boldsymbol{R_i \to R_i + kR_j}$.

EXAMPLE Let $A = \begin{bmatrix} 2 & -1 & 5 \\ -3 & 4 & \sqrt{2} \\ 7 & 6 & 3 \end{bmatrix}$.

$$\text{Applying } R_1 \to R_1 + 2R_3, \text{ we get } A \sim \begin{bmatrix} 16 & 11 & 11 \\ -3 & 4 & \sqrt{2} \\ 7 & 6 & 3 \end{bmatrix}.$$

THREE ELEMENTARY COLUMN OPERATIONS

(i) Interchange of any two columns The interchange of *i*th and *j*th columns is denoted by $\boldsymbol{C_i \leftrightarrow C_j}$.

EXAMPLE Let $A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 5 & 4 \\ 6 & 3 & \frac{1}{2} \end{bmatrix}$.

$$\text{Applying } C_1 \leftrightarrow C_2, \text{ we get } A \sim \begin{bmatrix} 1 & 2 & -3 \\ 5 & -1 & 4 \\ 3 & 6 & \frac{1}{2} \end{bmatrix}.$$

(ii) Multiplying each element of a column by a nonzero number Suppose each element of *i*th column of matrix *A* is multiplied by a nonzero number *k*.

Then, we write, $\boldsymbol{C_i \to kC_i}$.

EXAMPLE $\quad \text{Let } A = \begin{bmatrix} 3 & 1 & -5 \\ \sqrt{2} & -2 & 4 \\ 6 & 2 & 8 \end{bmatrix}.$

$$\text{Applying } C_3 \to 2C_3, \text{ we get } A \sim \begin{bmatrix} 3 & 1 & -10 \\ \sqrt{2} & -2 & 8 \\ 6 & 2 & 16 \end{bmatrix}.$$

(iii) Multiplying each element of a column of a given matrix *A* by a nonzero number and then adding to the corresponding elements of another column

Suppose each element of *j*th column of a given matrix *A* is multiplied by a nonzero number *k* and then added to the corresponding elements of *i*th column.

Then, we write, $\boldsymbol{C_i \to C_i + kC_j}$.

EXAMPLE $\quad \text{Let } A = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 1 \\ 5 & -2 & 6 \end{bmatrix}.$

$$\text{Applying } C_3 \to C_3 + 2C_1, \text{ we get } A \sim \begin{bmatrix} 2 & 0 & 8 \\ -1 & 3 & -1 \\ 5 & -2 & 16 \end{bmatrix}.$$

INVERTIBLE MATRICES *A square matrix A of order n is said to be invertible if there exists a square matrix B of order n such that*

$$\boldsymbol{AB = BA = I}.$$

Also, then B is called the inverse of A and we write, $\boldsymbol{A^{-1} = B}$.

EXAMPLE $\quad \text{Let } A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}. \text{ Then,}$

$$AB = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$BA = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$\therefore \quad AB = BA = I.$

Hence, $A^{-1} = B.$

THEOREM 1 (Uniqueness of Inverse) *Every invertible square matrix has a unique inverse.*

PROOF Let A be an invertible square matrix of order n.

If possible, let B as well as C be the inverse of A.

Then, $AB = BA = I$ and $AC = CA = I.$

Now, $AC = I \Rightarrow B(AC) = B \cdot I = B,$

$BA = I \Rightarrow (BA)C = I \cdot C = C.$

But, $B(AC) = (BA)C$ [by associative law of multiplication]

$\therefore \quad B = C.$

Hence, an invertible matrix has a unique inverse.

INVERSE OF A MATRIX BY ELEMENTARY ROW OPERATIONS

Let A be a square matrix of order n.

We can write, $A = I \cdot A.$ … (i)

Now, let a sequence of elementary row operations reduce $\boldsymbol{A}$ on LHS of (i) to I and I on RHS of (i) to a matrix B.

Then, $I = BA \Rightarrow I \cdot A^{-1} = (BA)A^{-1} = B(AA^{-1}) = BI$

$\Rightarrow A^{-1} = B.$

We can summarise the above method as given below.

METHOD

Step 1. Write $A = I \cdot A.$

Step 2. By using elementary row operations on A, transform it into a unit matrix.

Step 3. In the same order we apply elementary operations on I to convert it into a matrix B.

Step 4. Then, $A^{-1} = B.$

REMARK If on applying one or more elementary row operations on A, we obtain all zeros in one or more rows, then we say that A^{-1} does not exist.

SOLVED EXAMPLES

EXAMPLE 1 *By using elementary row operations, find the inverse of the matrix* $A = \begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix}.$

SOLUTION We have

$$\begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \frac{-1}{2} \end{bmatrix} \cdot A \qquad \left[R_2 \to \left(\frac{-1}{2}\right) R_2\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -\frac{1}{2} \end{bmatrix} \cdot A \qquad [R_1 \to R_1 + 2R_2].$$

Hence, $A^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & -\frac{1}{2} \end{bmatrix}$.

EXAMPLE 2 *By using elementary row operations, find the inverse of the matrix*

$$A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}.$$

SOLUTION We have

$$A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \to R_1 + R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \to (-1) \cdot R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} \cdot A \quad [R_2 \to R_2 + 4R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & \frac{3}{2} \end{bmatrix} \cdot A \quad \left[R_2 \to \left(\frac{-1}{2}\right) R_2\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \cdot A \qquad [R_1 \to R_1 + R_2].$$

Hence, $A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$.

EXAMPLE 3 *If* $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$, *show that* A^{-1} *does not exist.*

SOLUTION We have

$$\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} \cdot A \qquad \left[R_1 \to \frac{1}{6} R_1\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \cdot A \qquad [R_2 \to R_2 + 2R_1].$$

Thus, we have all zeros in second row of the left-hand side matrix. Hence, A^{-1} does not exist.

EXAMPLE 4 *By using elementary row operations, find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}.$$

SOLUTION We have

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot A \qquad \begin{bmatrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot A \qquad [R_2 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & -1 & 4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -2 & 0 & 1 \\ -15 & 1 & 9 \end{bmatrix} \cdot A \qquad \begin{bmatrix} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 + 9R_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -15 & 1 & 9 \end{bmatrix} \cdot A \qquad [R_2 \to (-1) \cdot R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \cdot A \left[R_3 \to \frac{1}{25} R_3 \right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 - 10R_3 \\ R_2 \to R_2 + 4R_3 \end{bmatrix}.$$

Hence, $A^{-1} = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$.

EXAMPLE 5 *By using elementary row operations, find the inverse of the matrix*

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}.$$

SOLUTION We have

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \to R_1 - R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & 0 \\ -3 & 3 & 1 \end{bmatrix} \cdot A \quad \left\{ R_2 \to \frac{1}{2} R_2 \right\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -5 & 6 & 1 \end{bmatrix} \cdot A \quad \left\{ \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix} \right\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} \cdot A \quad \left\{ R_3 \to \frac{1}{4} R_3 \right\}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-5}{8} & \frac{5}{4} & \frac{1}{8} \\ \frac{-3}{8} & \frac{3}{4} & \frac{-1}{8} \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} \cdot A. \quad \left\{ \begin{matrix} R_1 \to R_1 + \frac{1}{2} R_3 \\ R_2 \to R_2 - \frac{1}{2} R_3 \end{matrix} \right\}$$

Hence, $A^{-1} = \begin{bmatrix} \frac{-5}{8} & \frac{5}{4} & \frac{1}{8} \\ \frac{-3}{8} & \frac{3}{4} & \frac{-1}{8} \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$.

EXAMPLE 6 *By using elementary row transformations, find the inverse of the matrix*

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

SOLUTION We have

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_1 \leftrightarrow R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \quad [R_2 \to R_2 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 0 \end{bmatrix} \cdot A \quad [R_2 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 5 & -2 & 2 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 3R_3 \end{bmatrix}.$$

Hence, $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

EXAMPLE 7 *If* $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix}$, *show that* A^{-1} *does not exist.*

SOLUTION We have

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot A \quad \begin{bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot A \quad [R_3 \to R_3 + R_2].$$

Thus, we have all zeros in 3rd row of the left-hand side matrix.
Hence, A^{-1} does not exist.

EXERCISE 5E

Using elementary row transformations, find the inverse of each of the following matrices:

1. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ **[CBSE 2007]** 2. $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ 5. $\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$ 6. $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ 9. $\begin{bmatrix} 3 & 0 & 2 \\ 1 & 5 & 9 \\ 6 & 4 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$ 11. $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ **[CBSE 2011]**

13. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ **[CBSE 2008]** 14. $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ **[CBSE 2009]** 15. $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **[CBSE 2012]**

ANSWERS (EXERCISE 5E)

1. $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$ **3.** $\frac{1}{17} \cdot \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

4. $\begin{bmatrix} \frac{5}{22} & \frac{3}{22} \\ \frac{-2}{11} & \frac{1}{11} \end{bmatrix}$ **5.** $\begin{bmatrix} \frac{1}{4} & 0 \\ \frac{-1}{10} & \frac{1}{5} \end{bmatrix}$ **6.** $\begin{bmatrix} \frac{-9}{2} & \frac{7}{3} \\ 4 & -3 \end{bmatrix}$

7. $\frac{1}{2} \cdot \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$ **8.** $\frac{-1}{5} \cdot \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ **9.** $\frac{-1}{55} \cdot \begin{bmatrix} -1 & 8 & -10 \\ 47 & 9 & -25 \\ -26 & -12 & 15 \end{bmatrix}$

10. $\frac{1}{67} \cdot \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$ **11.** $\frac{-1}{8} \cdot \begin{bmatrix} 5 & -10 & -1 \\ 3 & -6 & 1 \\ 10 & -12 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

13. $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ **14.** $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$

EXERCISE 5F

Very-Short-Answer Questions

1. Construct a 3×2 matrix whose elements are given by
$$a_{ij} = \frac{1}{2}(i - 2j)^2.$$

2. Construct a 2×3 matrix whose elements are given by
$$a_{ij} = \frac{1}{2}|-3i + j|.$$

3. If $\begin{bmatrix} x + 2y & -y \\ 3x & 4 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & 4 \end{bmatrix}$, find the values of x and y. **[CBSE 2008C]**

4. Find the values of x and y, if
$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}.$$ **[CBSE 2008]**

5. If $x \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

6. If $\begin{bmatrix} x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, find the values of x, y, z, w.

7. If $\begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x + y \\ z + w & 3 \end{bmatrix} = 3\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, find the values of x, y, z, w.

8. If $A = \text{diag}\,(3 \quad -2 \quad 5)$ and $B = \text{diag}\,(1 \quad 3 \quad -4)$, find $(A + B)$.

9. Show that $\cos\theta \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \cdot \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = I$.

10. If $A = \begin{bmatrix} 1 & -5 \\ -3 & 2 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ -2 & 3 \end{bmatrix}$, find the matrix C such that $A + B + C$ is a zero matrix.

11. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ then find the least value of α for which $A + A' = I$.

12. Find the values of x and y for which
$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$ **[CBSE 2003]**

13. Find the values of x and y for which
$$\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$ **[CBSE 2003C]**

14. If $A = \begin{bmatrix} 4 & 5 \\ 1 & 8 \end{bmatrix}$, show that $(A + A')$ is symmetric. **[CBSE 2001]**

15. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, show that $(A - A')$ is skew-symmetric. **[CBSE 2001]**

16. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$, find a matrix X such that
$$A + 2B + X = O.$$

17. If $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$, find a matrix X such that
$$3A - 2B + X = O.$$ **[CBSE 2000]**

18. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A'A = I$.

19. If A and B are symmetric matrices of the same order, show that $(AB - BA)$ is a skew-symmetric matrix.

20. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 1$, find $f(A)$.

21. If the matrix A is both symmetric and skew-symmetric, show that A is a zero matrix.

ANSWERS (EXERCISE 5F)

1. $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$ **3.** $x = 2, y = -3$ **4.** $x = 3, y = 3$

5. $x = 3, y = -4$ **6.** $x = 3, y = 7, z = -2, w = 14$

7. $x = 2, y = 4, z = 1, w = 3$ **8.** diag (4 1 1)

10. $C = \begin{bmatrix} -4 & 4 \\ 1 & -1 \\ -2 & -1 \end{bmatrix}$ **11.** $\alpha = \frac{\pi}{3}$ **12.** $x = 2, y = 1$

13. $x = 1, y = 1$ **16.** $X = \begin{bmatrix} 0 & -1 \\ -4 & -11 \end{bmatrix}$

17. $X = \begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$ **20.** $f(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$